Stat 5101 Final Exam **December 15, 2012**

Name ______________________ Student ID ______________

The exam is closed book and closed notes. You may use three $8\frac{1}{2} \times 11$ sheets of paper with formulas, etc. You may also use the handouts on "brand name distributions" and Greek letters. Put all of your work on this test form (use the back if necessary). Show your work or give an explanation of your answer. No credit for numbers with no indication of where they came from.

The points for the questions total to 200. There are 8 pages and 8 problems.

1. [25 pts.] Suppose

$$\begin{aligned} E(Y \mid X) &= X \\ \operatorname{var}(Y \mid X) &= X \end{aligned}$$

and the marginal distribution of X is $\text{Gam}(\alpha, \lambda)$.

(a) Find $E(Y)$.

(b) Find $\operatorname{var}(Y)$.

2. [25 pts.] Define

$$f(x) = c \cdot \frac{1 + |x| + \cos(x)}{1 + x^4}, \qquad -\infty < x < \infty.$$

(a) Show that there exists a constant c such that f is a PDF.

(b) If X is a random variable having this PDF, for what positive real numbers β does $E(|X|^\beta)$ exist?

3. [25 pts.] Suppose X_1, X_2, $\ldots$ are IID Geo(p) random variables and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad (*)$$

What is the approximate normal distribution of

$$\overline{X}_n(1 + \overline{X}_n)$$

when n is large?

4. [25 pts.] Suppose X_1, X_2, $\ldots$ are IID $\mathcal{N}(\theta, \theta^4)$ random variables. What is the variance stabilizing transformation: for what function g does $g(\overline{X}_n)$ have approximate nondegenerate normal distribution for large n with variance that is a constant function of the parameter θ? As usual, $\overline{X}_n$ is defined by equation $(*)$ in problem 3.

5. [25 pts.]

(a) Suppose $\mathbf{X}$ has PDF

$$f(x) = \begin{cases} 0, & x < -1 \\ 3(1-x^2)/4 & -1 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

what is the DF of X? Be sure to define your function on the whole real line.

(b) Suppose $\mathbf{X}$ has DF (not PDF)

$$F(x) = \begin{cases} 0, & x < 0 \\ \sqrt{1-(1-x)^2} & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

what is the PDF of X? Define your function on the whole real line.

6. [25 pts.] Suppose X is an $\mathrm{Gam}(\alpha, \lambda)$ random variable. Find the PDF of the random variable $Y = X/(1+X)$. The definition of a function describes the domain as well as the rule.

7. [25 pts.] Suppose the random vector (X, Y) has the PDF

$$f(x,y) = \frac{x + y + \theta xy}{1 + \theta/4}, \qquad 0 < x < 1,\ 0 < y < 1,$$

where $\theta > 0$ is a parameter.

(a) Find the conditional PDF of Y given X. The definition of a function describes the domain as well as the rule.

(b) Find the conditional expectation of Y given X.

8. [25 pts.] Suppose the conditional distribution of Y given X has PDF

$$f(y \mid x) = x y^{-x-1}, \qquad 1 < y < \infty,$$

and suppose the marginal distribution of X is $\text{Exp}(\lambda)$. What is the conditional distribution of X given Y? Since this is a brand name distribution, no integrals need be done, it is enough to name the distribution and give its parameters as a function of y and λ. (Hint: laws of exponents and logarithms.)