# Stat 5101 Notes: Algorithms (thru 1st midterm)

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## Contents



# 1 Calculating an Expectation or a Probability

Probability is a special case of expectation (deck 1, slide 62).

### 1.1 From a PMF

If $f$ is a PMF having domain $S$ (the *sample space*) and $g$ is any function, then

$$E\{g(X)\} = \sum_{x \in S} g(x) f(x)$$

(deck 1, slide 56), and for any event $A$ (a subset of $S$)

$$\begin{aligned} \Pr(A) &= \sum_{x \in S} I_A(x) f(x) \\ &= \sum_{x \in A} f(x) \end{aligned}$$

(deck 1, slide 62).

### 1.2 From given Expectations using Uncorrelated

If $X$ and $Y$ are uncorrelated random variables, then

$$E(XY) = E(X)E(Y)$$

(deck 2, slide 73).

### 1.3 From given Expectations using Independent

If $X$ and $Y$ are independent random variables and $g$ and $h$ are any functions, then

$$E\{g(X)h(Y)\} = E\{g(X)\}E\{h(Y)\}$$

(deck 2, slide 76).

More generally, if $X_1$, $X_2$, $\ldots$, $X_n$ are independent random variables and $g_1$, $g_2$, $\ldots$, $g_n$ are any functions, then

$$E\left\{\prod_{i=1}^{n} g_i(X_i)\right\} = \prod_{i=1}^{n} E\{g_i(X_i)\}$$

(deck 2, slide 76).

## 1.4 From given Expectations using Linearity of Expectation

### 1.4.1 Expectation of Sum and Average

If $X_1$, $X_2$, …, $X_n$ are random variables, then

$$E\left\{\sum_{i=1}^{n} X_i\right\} = \sum_{i=1}^{n} E(X_i)$$

(deck 2, slide 10). In particular, if $X_1$, $X_2$, …, $X_n$ all have the same expectation $\mu$, then

$$E\left\{\sum_{i=1}^{n} X_i\right\} = n\mu$$

(deck 2, slide 90), and

$$E(\overline{X}_n) = \mu,$$

where

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{1}$$

(deck 2, slide 90).

### 1.4.2 Variance of Sum and Average

If $X_1$, $X_2$, …, $X_n$ are random variables, then

$$\begin{aligned}
\operatorname{var}\left\{\sum_{i=1}^{n} X_i\right\} &= \sum_{i=1}^{n}\sum_{j=1}^{n} \operatorname{cov}(X_i, X_j) \\
&= \sum_{i=1}^{n} \operatorname{var}(X_i) + \sum_{i=1}^{n}\sum_{j=1 j\neq i}^{n} \operatorname{cov}(X_i, X_j) \\
&= \sum_{i=1}^{n} \operatorname{var}(X_i) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \operatorname{cov}(X_i, X_j)
\end{aligned}$$

(deck 2, slide 71). In particular, if $X_1$, $X_2$, …, $X_n$ are uncorrelated, then

$$\operatorname{var}\left\{\sum_{i=1}^{n} X_i\right\} = \sum_{i=1}^{n} \operatorname{var}(X_i)$$

(deck 2, slide 75). More particular, if $X_1$, $X_2$, …, $X_n$ are uncorrelated and all have the same variance $\sigma^2$, then

$$\operatorname{var}\left\{\sum_{i=1}^{n} X_i\right\} = n\sigma^2$$

(deck 2, slide 90) and

$$\operatorname{var}(\overline{X}_n) = \frac{\sigma^2}{n}$$

(deck 2, slide 90), where $\overline{X}_n$ is given by (1).

#### 1.4.3 Expectation and Variance of Linear Transformation

If $X$ is a random variable and $a$ and $b$ are constants, then

$$\begin{aligned} E(a + bX) &= a + bE(X) \\ \operatorname{var}(a + bX) &= b^2 \operatorname{var}(X) \end{aligned}$$

(deck 2, slide 8 and slide 37).

#### 1.4.4 Covariance of Linear Transformations

If $X$ and $Y$ are random variables and $a$, $b$, $c$, and $d$ are constants, then

$$\operatorname{cov}(a + bX, c + dY) = bd \operatorname{cov}(X, Y)$$

(homework problem 3-7).

#### 1.4.5 Expectation and Variance of Vector Linear Transformation

If $\mathbf{X}$ is a random vector, $\mathbf{a}$ is a constant vector, and $\mathbf{B}$ is a constant matrix such that $\mathbf{a} + \mathbf{B}\mathbf{X}$ makes sense (the dimension of $\mathbf{a}$ and the row dimension of $\mathbf{B}$ are the same, and the dimension of $\mathbf{X}$ and the column dimension of $\mathbf{B}$ are the same), then

$$\begin{aligned} E(\mathbf{a} + \mathbf{B}\mathbf{X}) &= \mathbf{a} + \mathbf{B}E(\mathbf{X}) \\ \operatorname{var}(\mathbf{a} + \mathbf{B}\mathbf{X}) &= \mathbf{B} \operatorname{var}(\mathbf{X}) \mathbf{B}^T \end{aligned}$$

(deck 2, slide 64).

#### 1.4.6 "Short Cut" Formulas

If $X$ is a random variable, then

$$\operatorname{var}(X) = E(X^2) - E(X)^2$$

(deck 2, slide 21).

If $X$ and $Y$ are random variables, then

$$\operatorname{cov}(X, Y) = E(XY) - E(X)E(Y)$$

(homework problem 3-7).

# 2 Change of Variable Formulas

## 2.1 Discrete Distributions

### 2.1.1 One-to-one Transformations

If $f_X$ is the PMF of the random variable $X$, if $Y = g(X)$, and $g$ is an invertible function with inverse function $h$ (that is, $X = h(Y)$), then

$$f_Y(y) = f_X[h(y)]$$

and the domain of $f_Y$ is the range of the function $g$ (the set of possible $Y$ values) (deck 2, slide 86).

### 2.1.2 Many-to-one Transformations

If $f_X$ is the PMF of the random variable $X$ having sample space $S$ and $Y = g(X)$, then

$$f_Y(y) = \sum_{\substack{x \in S \\ g(x) = y}} f_X(x)$$

and the domain of $f_Y$ is the codomain of the function $g$ (deck 2, slide 81).

# 3 PMF and Independence

If $X_1$, …, $X_n$ are independent random variables having PMF $f_1$, …, $f_n$, then

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i)$$

(deck 1, slide 98).

In particular, if $X_1$, ..., $X_n$ are independent and identically distributed random variables having PMF $h$, then

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} h(x_i).$$