Stat 5101 Final Exam **December 18, 2009**

Name ______________________________ Student ID ____________________

The exam is closed book and closed notes. You may use three $8\frac{1}{2} \times 11$ sheets of paper with formulas, etc. You may also use the handouts on "brand name distributions" and Greek letters. Put all of your work on this test form (use the back if necessary). Show your work or give an explanation of your answer. No credit for numbers with no indication of where they came from.

The points for the questions total to 200. There are 8 pages and 8 problems.

1. [25 pts.] Suppose the conditional distribution of Y given X is exponential with mean X, and suppose the marginal distribution of X has mean μ and variance σ^2.

   (a) Find $E(Y)$.

   (b) Find $\operatorname{var}(Y)$.

2. [25 pts.] Define

$$f(x) = c \cdot \frac{1 + \cos(x)}{1 + \cos(x) + x^4}, \qquad -\infty < x < \infty.$$

(a) Show that there exists a constant c such that f is a PDF.

(b) If X is a random variable having this PDF, for what positive real numbers β does $E(|X|^\beta)$ exist?

3. [25 pts.] Suppose X_1, X_2, ... are IID $\text{Gam}(\alpha, \lambda)$ random variables and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad (*)$$

What is the approximate normal distribution of

$$\frac{\overline{X}_n}{1 + \overline{X}_n}$$

when n is large?

4. [25 pts.] Suppose X_1, X_2, $\ldots$ are IID $\text{Gam}(3.33, \lambda)$ random variables. What is the variance stabilizing transformation: for what function g does $g(\overline{X}_n)$ have approximate normal distribution for large n with variance that is a constant function of the parameter λ? As usual, $\overline{X}_n$ is defined by equation $(*)$ in problem 3.

5. [25 pts.] Suppose X_1, ..., X_n is a sequence of exchangeable random variables with

$$
\begin{aligned}
E(X_i) &= \mu \\
\operatorname{var}(X_i) &= \sigma^2 \\
\operatorname{cov}(X_i, X_j) &= \rho\sigma^2, \qquad i \neq j
\end{aligned}
$$

As usual, let $\overline{X}_n$ be defined by equation $(*)$ in problem 3.

(a) Find the mean of the random variable $\overline{X}_n$.

(b) Find the variance of the random variable $\overline{X}_n$.

6. [25 pts.] Suppose X is an $\text{Exp}(\lambda)$ random variable. Find the PDF of the random variable $Y = 1/(1+X)$. The definition of a function describes the domain as well as the rule.

7. [25 pts.] Suppose the random vector (X, Y) has the PDF

$$f(x, y) = \frac{1}{3}(x + y^2)e^{-x-y}, \qquad 0 < x < \infty,\ 0 < y < \infty.$$

Find the conditional expectation (note: expectation not PDF) of Y given X. The definition of a function describes the domain as well as the rule.

8. [25 pts.] Suppose the conditional distribution of Y given X is $\mathcal{N}(\mu, 1/X)$, where μ is a known real number, and suppose the marginal distribution of X is $\text{Gam}(\alpha, \lambda)$. What is the conditional distribution of X given Y? Since this is a brand name distribution, no integrals need be done, it is enough to name the distribution and give its parameters as a function of y, α, λ, and μ.