

Stat 5101 (Geyer) Fall 2009

Homework Assignment 4

Due Wednesday, October 7, 2009

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

**4-1.** If  $U$ ,  $V$ ,  $X$ , and  $Y$  are any random variables, show that

$$\text{cov}(U + V, X + Y) = \text{cov}(U, X) + \text{cov}(V, X) + \text{cov}(U, Y) + \text{cov}(V, Y)$$

**4-2.** Suppose  $X_1, X_2, X_3$  are IID with mean  $\mu$  and variance  $\sigma^2$ . Calculate the mean vector and variance matrix of the random vector

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} X_1 - X_2 \\ X_2 - X_3 \\ X_3 - X_1 \end{pmatrix}$$

**4-3.** Suppose  $X$  and  $Y$  are independent random variables, with means  $\mu_X$  and  $\mu_Y$ , respectively, and variances  $\sigma_X^2$  and  $\sigma_Y^2$ , respectively. Calculate

$$E(X^2Y^2)$$

in terms of  $\mu_X$ ,  $\mu_Y$ ,  $\sigma_X^2$ , and  $\sigma_Y^2$ .

**4-4.** Suppose 6 balls that are indistinguishable except for color are placed in an urn and suppose 3 balls are red and 3 are white. Suppose 2 balls are drawn. What is the probability the one is red and the other white under each of the following conditions?

- (a) The 2 balls constitute a random sample with replacement from the urn.
- (b) The 2 balls constitute a random sample without replacement from the urn.

What is the probability the both balls are red under each of the following conditions?

- (c) The 2 balls constitute a random sample with replacement from the urn.
- (d) The 2 balls constitute a random sample without replacement from the urn.

**4-5.** If  $X_1, \dots, X_n$  are exchangeable random variables, show that

$$\text{cov}(X_1, X_2) \geq -\frac{\text{var}(X_1)}{n-1}$$

Hint: consider the variance of  $X_1 + \dots + X_n$ .

**4-6.** Suppose  $X_1, X_2, \dots$  are IID random variables having mean  $\mu$  and variance  $\tau^2$ . For each  $i \geq 1$  define

$$Y_i = \sum_{j=1}^5 X_{i+j}$$

Then  $Y_1, Y_2, \dots$  is called a *moving average of order 5* time series, MA(5) for short. It is a weakly stationary time series.

- (a) Calculate  $E(Y_i)$ .
- (b) Calculate  $\text{var}(Y_i)$ .
- (c) Calculate  $\text{cov}(Y_i, Y_{i+k})$ , for  $k = 1, 2, \dots$ .

**4-7.** Suppose  $X_1$  and  $X_2$  are IID random variables that are uniformly distributed on the set  $\{1, 2, 3, 4, 5\}$ .

- (a) Find the PMF of the random vector  $\mathbf{Y} = (X_1, X_1 + X_2)$ .
- (b) Are the components of  $\mathbf{Y}$  independent?
- (c) Are the components of  $\mathbf{Y}$  uncorrelated?

**4-8.** Suppose  $X_1, \dots, X_k$  are independent binomial random variables with different sample sizes but the same success probability, say  $X_i$  has the  $\text{Bin}(n_i, p)$  distribution. Show that  $Y = X_1 + \dots + X_k$  has the  $\text{Bin}(n_1 + \dots + n_k, p)$  distribution. Hint: no calculation necessary. This follows from something we already know.

## Review Problems from Previous Midterm

**4-9.** Suppose  $\mathbf{X}$  is a random vector having mean vector

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

and variance matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

- (a) Calculate  $E(X_1 + 2X_2 + 3X_3)$ .
- (b) Calculate  $\text{var}(X_1 + 2X_2 + 3X_3)$ .

**4-10.** Suppose a basketball player is shooting free throws and the results (success or failure) are considered to be independent and identically distributed Bernoulli random variables with success probability  $3/4$  (code success as one and failure as zero). The player shoots three free throws. What is the probability that she makes at least one?