Stat 5101 Final Exam **December 17, 2008**

Name ______________________ Student ID ______________

The exam is closed book and closed notes. You may use three $8\frac{1}{2} \times 11$ sheets of paper with formulas, etc. You may also use the handouts on "brand name distributions" and Greek letters. Put all of your work on this test form (use the back if necessary). Show your work or give an explanation of your answer. No credit for numbers with no indication of where they came from.

The points for the questions total to 200. There are 7 pages and 10 problems.

1. [20 pts.] Suppose the random vector (X, Y) has the PDF

$$f(x, y) = \frac{1}{2}(x + y + xy)e^{-x}, \qquad 0 < x < \infty,\ 0 < y < 1. \tag{1}$$

Find the marginal PDF of Y. The definition of a function describes the domain as well as the rule.

2. [20 pts.] Suppose the random vector (X, Y) has the PDF

$$f(x, y) = \frac{6}{7}(x + y)^2, \qquad 0 < x < 1,\ 0 < y < 1.$$

Find the conditional PDF of Y given X. The definition of a function describes the domain as well as the rule.

3. [20 pts.] Suppose the conditional distribution of Y given X is $\text{Gam}(\beta, X)$, where β is a known real number, and suppose the marginal distribution of X is $\text{Gam}(\alpha, \lambda)$. What is the conditional distribution of X given Y? Since this is a brand name distribution, no integrals need be done, it is enough to name the distribution and give its parameters as a function of Y, α, β, and λ.

4. [20 pts.] Suppose X_1, …, X_N are IID having mean μ and variance σ^2 where N is a Geo(p) random variable independent of all of the X_i. Let

$$Y = \sum_{i=1}^{N} X_i,$$

with the convention that $N = 0$ implies $Y = 0$.

(a) Find $E(Y)$.

(b) Find $\operatorname{var}(Y)$.

5. [20 pts.] Suppose telephone calls arriving at a call center between 10:00 am and 11:00 am on a weekday can be modeled as a Poisson process with rate 10 calls per minute.

   (a) What is the mean and standard deviation of the number of the number of calls arriving in a 20 minute interval?

   (b) What is the mean and standard deviation of the time interval between now and the time of arrival of the next call to arrive?

   (c) What is the mean and standard deviation of the time interval between now and the time of arrival of the fifth call to arrive (after now)?

6. [20 pts.] Suppose X_1, X_2, $\ldots$ are IID Geo(p) random variables and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

What is the approximate normal distribution of

$$\frac{1}{1 + \overline{X}_n}$$

when n is large?

7. [20 pts.] Suppose X is a chi$^2(n)$ random variable. What is the variance stabilizing transformation: for what function g does $g(X)$ have approximate normal distribution for large n with variance that is a constant function of the parameter n?

8. [20 pts.] Define

$$h(x) = \frac{x}{1 + \sqrt{x} + x^5}, \qquad x > 0$$

(a) Show that there exists a constant c such that $x \mapsto c \cdot h(x)$ is a PDF.

(b) If X is a random variable having this PDF, for what positive real numbers β does $E(X^\beta)$ exist?

9. [20 pts.] Calculate the DF corresponding to the PDF

$$f(x) = \frac{3(1 - x^2)}{4}, \qquad -1 < x < 1$$

Define the DF on the whole real line.

10. [20 pts.] Suppose $\mathbf{X}$ is a random vector having mean vector

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

and variance matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

Let

$$Y = X_1 + X_2 + X_3$$

(a) Calculate $E(Y)$

(b) Calculate $\operatorname{var}(Y)$.