Stat 5101 (Geyer) Fall 2008
Homework Assignment 12
Due Wednesday, December 10, 2008

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

12-1. Give the details of the argument that the $\text{Poi}(\mu)$ distribution is approximately normal when μ is large.

12-2. Suppose X_1, X_2, ... are IID with mean μ and variance σ^2 and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

What is the approximate normal distribution of $\sin(\overline{X}_n)$ when n is large?

12-3. Suppose X_1, X_2, ... are IID $\text{Poi}(\mu)$ random variables and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

To what random variable does

$$\sqrt{n}\bigl(e^{-\overline{X}_n} - e^{-\mu}\bigr)$$

converge in distribution?

12-4. Suppose X_1, X_2, ... are IID $\text{Ber}(p)$ random variables with $0 < p < 1$ and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

(a) What is the approximate normal distribution of $\overline{X}_n(1 - \overline{X}_n)$ when n is large?

(b) There is something unusual about the case $p = 1/2$. What is that?

12-5. Suppose X is a $\text{Poi}(\mu)$ random variable. For what function g does $g(X)$ have approximate normal distribution for large μ with variance that is a constant function of the parameter?

12-6. Suppose X_1, X_2, $\ldots$ are IID Exp(λ) random variables and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

For what function g does $g(\overline{X}_n)$ have approximate normal distribution for large n with variance that is a constant function of the parameter?

12-7. Show that the $\mathcal{N}\big(n\mathbf{p}, n(\mathbf{P} - \mathbf{p}\mathbf{p}^T)\big)$ distribution is degenerate: if $\mathbf{X} = (X_1 + \cdots + X_k)$ is a random vector having this distribution, then

$$X_1 + \cdots + X_k = n$$

almost surely. Here $\mathbf{P} - \mathbf{p}\mathbf{p}^T$ is the variance matrix of the Multi$(n, \mathbf{p})$ distribution.
Hint: If you want to use matrix notation in this problem, define the vector $\mathbf{u} = (1, 1, \ldots, 1)$ so that

$$\mathbf{u}^T \mathbf{X} = X_1 + \cdots + X_k$$

12-8. Suppose X_1, X_2, $\ldots$, is an IID sequence of random variables, having first four ordinary moments

$$\alpha_i = E(X_n^i), \qquad i = 1, \ldots, 4.$$

Define

$$Y_n = X_n^2, \qquad n = 1, 2, \ldots$$

and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$
$$\overline{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i$$

What is the approximate normal distribution of $\overline{Y}_n - \overline{X}_n^2$ when n is large?
Hint: Slides 87–89, deck 6 and the multivariate delta method.