Stat 5101 (Geyer) Fall 2008
Homework Assignment 10
Due Wednesday, November 26, 2008

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

10-1. Suppose the random vector $(X_1, \ldots, X_k)$ has the $\text{Multi}(n, \mathbf{p})$ distribution and $i \neq j$.

(a) Find $E(X_i - X_j)$.

(b) Find $\text{var}(X_i - X_j)$.

10-2. Suppose the random vector $(X_1, \ldots, X_k)$ has a multinomial distribution, and suppose we "factor" the joint PMF as the product of univariate marginals and conditionals

$$f(x_1, \ldots, x_k) = f(x_1 \mid x_2 \ldots, x_k) f(x_2 \mid x_3 \ldots, x_k) \cdots f(x_{k-1} \mid x_k) f(x_k)$$

Show that each of the univariate conditionals except for $f(x_1 \mid x_2 \ldots x_k)$ is binomial and the univariate marginal is binomial. Give the parameters of each in terms of the parameters of the joint distribution. What is the distribution of X_1 given X_2, …, X_k?

Hint: no calculation necessary. All of the answer is determined by theory on the slides.

10-3. Suppose X_1, …, X_k are independent Poisson random vectors. Show that the conditional distribution of the random vector $(X_1, \ldots, X_k)$ given $X_1 + \ldots + X_k$ is multinomial. Give the parameters of this multinomial in terms of the parameters of the Poisson distributions.

10-4. Suppose the marginal distribution of the random variable N is Poisson having mean $\mu > 0$, and suppose the conditional distribution of the random vector $(X_1, \ldots, X_k)$ given N is $\text{Multi}(N, \mathbf{p})$. Show that the marginal distribution of $(X_1, \ldots, X_k)$ has independent components and that the distribution of each component is Poisson. Calculate $E(X_i)$, which is the parameter of its marginal Poisson distribution.

Hint: the degeneracy of the conditional distribution of $(X_1, \ldots, X_k)$ given N makes this tricky. Eliminate X_1 by writing it as a function of X_2, …, X_k and N. Then write the joint distribution of X_2, …, X_k and N as the product of marginal and conditional. Now eliminate N by writing it as a function of X_1, …, X_k to obtain the joint PMF of X_1, …, X_k.

10-5. Suppose (X_1, X_2) is a bivariate normal random vector, and assume it is nondegenerate. Write its PDF in terms of the mean vector and variance matrix. Then rewrite its PDF in terms of new parameters, which are

$$\begin{aligned} \mu_1 &= E(X_1) \\ \mu_2 &= E(X_2) \\ \sigma_1 &= \operatorname{sd}(X_1) \\ \sigma_2 &= \operatorname{sd}(X_2) \\ \rho &= \operatorname{cor}(X_1, X_2) \end{aligned}$$

Then simplify your expression for the PDF so it contains no matrices, no matrix inverses, determinants, or matrix multiplication.

Hint: the inverse of a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

can be done by Cramer's rule obtaining

$$\mathbf{A}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}$$

assuming $\mathbf{A}$ is invertible, which it is if $\det(\mathbf{A})$ is not zero.

10-6. Suppose (X_1, X_2) is a nondegenerate bivariate normal random vector. Calculate the conditional PDF of X_1 given X_2 not using the theory developed in class. Just use conditional = joint/marginal, where the joint is the PDF you found in problem 10-5.

10-7. Suppose (X_1, X_2) is a nondegenerate bivariate normal random vector.

(a) What is the best prediction of X_1 that is a function of X_2 when "best" means minimizing expected squared prediction error?

(b) What is the best prediction of X_1 that is a function of X_2 when "best" means minimizing expected absolute prediction error?

10-8. If $\mathbf{X}$ is a standard normal random vector and $\mathbf{O}$ is an orthogonal matrix of the same dimension, show that $\mathbf{O}^T \mathbf{X}$ is a standard normal random vector.

10-9. Suppose $\mathbf{Y}$ is a nondegenerate multivariate normal random vector having mean vector $\boldsymbol{\mu}$ and variance matrix $\mathbf{M}$. Show that the random scalar

$$(\mathbf{Y} - \boldsymbol{\mu})^T \mathbf{M}^{-1} (\mathbf{Y} - \boldsymbol{\mu})$$

has the chi-square distribution with degrees of freedom that is the dimension of $\mathbf{Y}$.

Hint: write $\mathbf{Y} = \boldsymbol{\mu} + \mathbf{M}^{1/2}\mathbf{X}$ where

$$\mathbf{M} = \mathbf{O}\mathbf{D}\mathbf{O}^T$$

is the spectral decomposition of $\mathbf{M}$ and $\mathbf{X}$ is a standard multivariate normal random vector. Then write $\mathbf{X}$ as a function of $\mathbf{Y}$, and consider the distribution of $\mathbf{X}^T\mathbf{X}$.