Stat 5101 (Geyer) Fall 2008
Homework Assignment 8
Due Wednesday, November 5, 2008

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

8-1. Suppose (X, Y) has the PDF

$$f(x, y) = \frac{1 + x}{\pi}, \qquad x^2 + y^2 < 1.$$

Find the PDF of the random vector (R, T) which is (X, Y) expressed in polar coordinates, that is,

$$\begin{aligned} X &= R\cos(T) \\ Y &= R\sin(T) \end{aligned}$$

8-2. In class, we found out that the $\text{Unif}(a, b)$ family of distributions is a location-scale family, but a and b are not the mean and variance. Suppose we wanted to parametrize the family so that the location parameter is the mean μ and the scale parameter is the standard deviation σ. Then what is the form of the PDFs of the family?

8-3. If X has the $\text{Beta}(\alpha_1, \alpha_2)$ distribution, show that $E\{X^{\beta_1}(1 - X)^{\beta_2}\}$ exists if and only if $\beta_1 > -\alpha_1$ and $\beta_2 > -\alpha_2$.

8-4. Let

$$f(x) = ce^{-|x|}, \qquad -\infty < x < \infty.$$

(a) Show that there exists a constant c such that f is a probability distribution and determine the c that makes it so.

(b) For what positive β does $E(|X|^\beta)$ exist?

(c) Find $E(X)$.

(d) Find $\text{var}(X)$.

(e) Write the PDFs of location-scale family containing f using the mean for the location parameter and the standard deviation for the scale parameter.

8-5. Suppose $X_1, \ldots, X_n$ are IID standard normal random variables, then

$$R^2 = \sum_{i=1}^{n} X_i^2$$

has the chi-square distribution on n degrees of freedom.

(a) Find is the PDF of R?

(b) Find $E(R)$.

(c) Find $E(R^2)$.

8-6. Let

$$f(x) = c \cdot \frac{1}{2 + \cos(x) + \cos(2x) + x^2 + x^6}, \qquad -\infty < x < \infty$$

(a) Show that there exists a constant c such that f is a probability distribution.

(b) For what positive real β does $E(|X|^\beta)$ exist?

Hint: $-1 \le \cos(x) \le 1$ for all x and $\cos(0) = 1$. You can't integrate f. You have to use the theory about existence.

8-7. Suppose $\alpha_1 > 0$ and $\alpha_2 > 0$. Let

$$f(x) = c \cdot \sin(x) x^{\alpha_1 - 1}(1 - x)^{\alpha_2 - 1}, \qquad 0 < x < 1.$$

(a) Show that there exists a constant c such that f is a probability distribution.

(b) For what real β_1 and β_2 does $E\{X^{\beta_1}(1 - X)^{\beta_2}\}$ exist?

Hint:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

and $0 \le \sin(x) \le 1$ for $0 \le x \le \pi$. You can't integrate f. You have to use the theory about existence.

8-8. Show that, if a random variable X is symmetric about a point a, then a is a median of X. Comment: we already know that if the mean exists, then a is a mean of X. Hence the mean, median, and center of symmetry are all the same for a symmetric distribution, if the mean exists.

8-9. If Φ and Φ^{-1} denote the distribution and quantile functions, respectively, for a standard normal random variable, what are the distribution and quantile functions for a $\mathcal{N}(\mu, \sigma^2)$ random variable?

8-10. Let X have the standard Cauchy distribution.

(a) Find the quantile function for X.

(b) Find the median of X.

(c) Find the lower and upper quartiles of X.

8-11. Suppose U is a $\text{Unif}(0, 1)$ random variable, and suppose F is a distribution function and G the corresponding quantile function. Show that the random variable $G(U)$ has F as its DF. Hint: for $0 < u < 1$ show that $G(u) \leq x$ if and only if $F(x) \geq u$. This requires use of the fact that DF are right continuous.

8-12. Suppose X has the $\text{Exp}(1)$ distribution.

(a) What is the best prediction of the value of X if minimizing expected squared error is the criterion?

(b) What is the best prediction of the value of X if minimizing expected absolute error is the criterion?

In this problem, we want numeric answers so we can see how different they are.

8-13. Suppose X has the $\text{Gam}(2, 1)$ distribution.

(a) What is the best prediction of the value of X if minimizing expected squared error is the criterion?

(b) What is the best prediction of the value of X if minimizing expected absolute error is the criterion?

In this problem, we want numeric answers so we can see how different they are. The answer to (b) can only be found using the computer.

8-14. Suppose the random vector (X, Y) has variance matrix

$$\begin{pmatrix} 4 & -3 \\ -3 & 9 \end{pmatrix}$$

Find $\text{sd}(X)$, $\text{sd}(Y)$, and $\text{cor}(X, Y)$. Note: $\text{cor}(X, Y)$ is correlation not covariance.

8-15. Suppose $\text{sd}(X) = 5$, $\text{sd}(Y) = 7$, and $\text{cor}(X, Y) = 2/3$. Find the variance matrix of the random vector (X, Y).