Stat 5101 (Geyer) Fall 2008
Homework Assignment 7
Due Wednesday, October 29, 2008

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

7-1. If X has the $\text{Gam}(\alpha, \lambda)$ distribution, we calculated in class that

$$E(X^\beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\lambda^\beta}.$$

(a) Find $E(X^2)$

(b) Find $\text{var}(X)$.

None of your answers should contain gamma functions (use the gamma function recursion formula to simplify).

7-2. If X has the $\text{Beta}(\alpha_1, \alpha_2)$ distribution, show that

$$E\{X^{\beta_1}(1 - X)^{\beta_2}\} = \frac{\Gamma(\alpha_1 + \alpha_2)\Gamma(\alpha_1 + \beta_1)\Gamma(\alpha_2 + \beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)}$$

Hint: use the fact that the PDF of the beta distribution integrates to one, just like we did for the gamma distribution. You may ignore the issue of when the integral exists (it exists when $\beta_1 > -\alpha_1$ and $\beta_2 > -\alpha_2$, but we don't know how to prove that yet).

7-3. Suppose X has the $\text{Beta}(\alpha_1, \alpha_2)$ distribution.

(a) Find $E(X)$.

(b) Find $E(X^2)$.

(c) Find $\text{var}(X)$.

None of your answers should contain gamma functions (use the gamma function recursion formula to simplify). Hint: use the result of problem 7-2.

7-4. If X and Y are independent gamma random variables with shape parameters α_1 and α_2, respectively, and the same rate parameter, then we proved in class that $V = X/(X + Y)$ has the $\text{Beta}(\alpha_1, \alpha_2)$ distribution. Suppose instead we were interested in the random variable $W = X/Y$.

(a) Express W as a function of V.

(b) Show that this function is invertible (that is, V is also a function of W).

(c) Find the PDF of W. Be sure to give the domain as well as the formula.

7-5. Suppose the random vector (X, Y) has the uniform distribution on the square $(0, 1)^2$. Find the PDF of the random vector (U, V) where

$$\begin{aligned} U &= X + Y \\ V &= \frac{X}{X + Y} \end{aligned}$$

Be sure to give the domain as well as the formula. Express the domain clearly enough so that you can draw a correct picture of it. Hint: the support of (U, V) is

$$\{ (u, v) \in \mathbb{R}^2 : 0 < uv < 1 \text{ and } 0 < u(1 - v) < 1 \}.$$

The problem is to express clearly how this constrains u and v.

7-6. Suppose X has a distribution that is symmetric about a point a. Show that every odd central moment is zero.

7-7. Show that the moment generating function of the standard normal distribution is $t \mapsto e^{t^2/2}$. Hint: use the fact that a general normal PDF integrates to one.

7-8. Suppose X has the standard normal distribution.

(a) Find $E(X^2)$.

(b) Find $E(X^4)$.

Hint: use the result of problem 7-7.

7-9. Suppose X has a distribution that is symmetric about a point a, and let $Y = \mu + \sigma X$ with $\sigma > 0$. Let $\mu_{X,k}$ and $\mu_{Y,k}$ denote the k-th central moments of X and Y, respectively. Show that

$$\mu_{Y,k} = \sigma^k \mu_{X,k}$$

7-10. Suppose X has the $\mathcal{N}(\mu, \sigma^2)$ distribution.

(a) Find $E(X)$.

(b) Find $\operatorname{var}(X)$.

(c) Find $E\{(X - \mu)^3\}$.

(d) Find $E\{(X - \mu)^4\}$.

Hint: use the results of problems 7-6, 7-8, and 7-9.