Stat 8112 Lecture Notes

The Weak Law of Large Numbers

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January 23, 2013

Feller (1971, p. 565) gives a very complete description of the weak law of large numbers (his Theorem of Section 2a of Chapter XVII), which we repeat here.

Theorem 1. *Suppose X_1, X_2, ... are independent and identically distributed random variables having characteristic function φ and distribution function F, and write*

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Then the following three conditions are equivalent (each implies the others)

(i) *φ is differentiable at zero and $\varphi'(0) = i\mu$.*

(ii) *As $t \to \infty$ both of the following hold*

$$t[1 - F(t) + F(-t)] \to 0 \tag{1a}$$

$$\int_{-t}^{t} x \, F(dx) \to \mu \tag{1b}$$

(iii) *$\overline{X}_n \xrightarrow{P} \mu$.*

The notation in (1b) means integration with respect to the probability measure that is the law of the X_i.

Here is an example. Define

$$F(t) = \begin{cases} 1 - \frac{\log(2)}{t \log(t)}, & t \geq 2 \\ \frac{1}{2}, & -2 \leq t \leq 2 \\ \frac{\log(2)}{|t| \log(|t|)}, & t \leq -2 \end{cases}$$

This clearly satisfies (1a). The fact that $F(-t) = 1 - F(t)$ implies the distribution is symmetric about zero. Hence the integral in (1b) is zero by symmetry for all t, and (1b) holds with $\mu = 0$. Thus, by the theorem, (i) and (iii) of the theorem also hold. In particular, the weak law of large numbers ((iii) of the theorem) holds.

The reason for choosing this example is that we can show the mean does not exist, hence, by Theorem 4(c) in Ferguson (1996) the strong law of large numbers does not hold.

The expectation of a nonnegative random variable Y having distribution function G can be calculated

$$E(Y) = \int_0^\infty [1 - G(t)] \, dt$$

(Fristedt and Gray, 1996, Proposition 19 of Chapter 4).

We use this to calculate $E(|X|)$ where X has the distribution function F. The distribution function of $|X|$ is

$$F_{|X|}(t) = 1 - 2[1 - F(t)], \qquad t \geq 0.$$

Hence

$$E(|X|) = \int_2^\infty \frac{2 \log(2)}{t \log(t)} \, dt.$$

This is finite if and only if the series

$$\sum_{n=2}^\infty \frac{1}{t \log(t)}$$

is summable. For that we use the Cauchy condensation test (Stromberg, 1981, Theorem 7.1), which says for a series having monotone decreasing positive terms a_n

$$\sum_{n=0}^\infty a_n \quad \text{and} \quad \sum_{k=0}^\infty 2^k a_{2^k}$$

either both converge or both diverge. So does the series with terms

$$\frac{2^k}{2^k \log(2^k)} = \frac{1}{k \log(2)}$$

converge? It does not because it is the harmonic series, which is well known not to converge, because

$$\sum_{k=1}^\infty \frac{1}{k} \qquad \text{and} \qquad \int_1^\infty \frac{dt}{t}$$

either both converge or both diverge, and the integral obviously diverges.

References

Feller, W. (1971). *An Introduction to Probability Theory and Its Applications,* 2nd edition, volume 2. New York: Wiley.

Ferguson, T. S. (1996). *A Course in Large Sample Theory.* London: Chapman & Hall.

Fristedt, B. E. and Gray, L. F. (1996). *A Modern Approach to Probability Theory.* Boston: Birkhäuser.

Stromberg, K. R. (1981). *Introduction to Classical Real Analysis.* Belmont, CA: Wadsworth.