

Name _____ Student ID _____

The exam is closed book and closed notes. You may use one $8\frac{1}{2} \times 11$ sheet of paper with formulas, etc. You may also use the handouts on “brand name distributions” and Greek letters. You may use a calculator.

Put all of your work on this test form (use the back if necessary). Show your work or give an explanation of your answer. No credit for answers with no indication of where they came from. Leave no undone integrals, expectations, or derivatives in your answers, but other than that requirement there is no unique “correct” simplification. Any correct answer gets full credit, except as explicitly stated in questions.

Abbreviations used: independent and identically distributed (IID), maximum likelihood estimate (MLE), probability density function (PDF), probability mass function (PMF).

The points for the questions total to 100. There are 7 pages and 5 problems.

1. [20 pts.] Suppose X_1, \dots, X_n are IID from the distribution having PDF

$$f_{\sigma}(x) = \sqrt{\frac{\sigma}{2\pi}} \cdot x^{-3/2} e^{-\sigma/(2x)}, \quad 0 < x < \infty,$$

where σ is an unknown parameter satisfying $0 < \sigma < \infty$.

(a) Find the log likelihood for σ .

(b) Find the MLE for σ .

(c) Show your MLE is the unique global maximizer of the log likelihood.

2. [20 pts.] Suppose X is $\text{Bin}(n, p)$, where $p = 2\theta - \theta^2$ and θ is an unknown parameter satisfying $0 \leq \theta \leq 1$. We have only the one observation X .

(a) Find the log likelihood for θ . **Hint:** writing

$$\begin{aligned} p &= \theta(2 - \theta) \\ 1 - p &= (1 - \theta)^2 \end{aligned}$$

simplifies the log likelihood a lot and makes the following parts easier.

(b) Find the MLE for θ . **Hint:** the roots of the quadratic polynomial $ax^2 + bx + c$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(c) Find the (expected) Fisher information for θ .

(d) Find the asymptotic distribution of your MLE for θ . The mean and variance of the asymptotic normal distribution should be expressed as functions of θ and n only.

3. [20 pts.] Suppose X_1, \dots, X_n are IID from the distribution having PDF

$$f(x | \sigma) = \sqrt{\frac{\sigma}{2\pi}} \cdot x^{-3/2} e^{-\sigma/(2x)}, \quad 0 < x < \infty,$$

where σ is an unknown parameter satisfying $0 < \sigma < \infty$ (this is the same distribution as in question 1). Suppose the prior distribution for σ is $\text{Gam}(\alpha, \lambda)$, where α and λ are hyperparameters satisfying $\alpha > 0$ and $\lambda > 0$. Find the posterior distribution for σ .

4. [20 pts.] Suppose X_1, \dots, X_n are IID $\text{Gam}(\alpha, \lambda)$, where α is known and λ is unknown. Suppose the prior distribution for λ is the improper prior with density function

$$g(\lambda) = \frac{1}{\lambda}, \quad 0 < \lambda < \infty,$$

(we can't say *probability* density function because this is an improper prior, so g is not a constant times a PDF).

- (a) Find the posterior distribution for λ .

- (b) For what values of the data x_1, \dots, x_n does your answer to part (a) make sense?

5. [20 pts.] Suppose X_1, \dots, X_n are IID $\mathcal{N}(\mu, \sigma^2)$ with both μ and σ^2 unknown parameters. Show that their joint distribution is an exponential family. Identify the natural parameter vector and the natural statistic vector.