Stat 5102 Notes: ARE of Method of Moments Estimators for the Poisson Distribution

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What is the ARE of the two method of moments estimators compared on slides 61–62. deck 2.

These are $\overline{X}_n$ and V_n considered as estimators of the mean of the Poisson distribution. The asymptotic distributions are

$$\begin{aligned}
\overline{X}_n &\approx \mathcal{N}\left(\mu, \frac{\mu}{n}\right) \\
V_n &\approx \mathcal{N}\left(\mu, \frac{\mu_4 - \mu^2}{n}\right)
\end{aligned}$$

In order to figure out the asymptotic variance of the latter we need to calculate the fourth central moment of the Poisson distribution. We start with the moment generating function.

$$\begin{aligned}
\varphi(t) &= E(e^{tX}) \\
&= \sum_{x=0}^{\infty} e^{xt} \frac{\mu^x}{x!} e^{-\mu} \\
&= \sum_{x=0}^{\infty} \frac{(e^t \mu)^x}{x!} e^{-\mu} \\
&= e^{\mu(e^t - 1)}
\end{aligned}$$

and this has derivatives

$$
\begin{aligned}
\varphi'(t) &= e^{\mu(e^t-1)}\mu e^t \\
\varphi''(t) &= e^{\mu(e^t-1)}(\mu e^t)^2 + e^{\mu(e^t-1)}\mu e^t \\
&= e^{\mu(e^t-1)}[\mu^2 e^{2t} + \mu e^t] \\
\varphi'''(t) &= e^{\mu(e^t-1)}[\mu^2 e^{2t} + \mu e^t]\mu e^t + e^{\mu(e^t-1)}[2\mu^2 e^{2t} + \mu e^t] \\
&= e^{\mu(e^t-1)}[\mu^3 e^{3t} + 3\mu^2 e^{2t} + \mu e^t] \\
\varphi''''(t) &= e^{\mu(e^t-1)}[\mu^3 e^{3t} + 3\mu^2 e^{2t} + \mu e^t]\mu e^t \\
&\qquad + e^{\mu(e^t-1)}[3\mu^3 e^{3t} + 6\mu^2 e^{2t} + \mu e^t] \\
&= e^{\mu(e^t-1)}[\mu^4 e^{4t} + 6\mu^3 e^{3t} + 7\mu^2 e^{2t} + \mu e^t]
\end{aligned}
$$

and this gives ordinary moments

$$
\begin{aligned}
\alpha_1 &= E(X) = \varphi'(0) = \mu \\
\alpha_2 &= E(X^2) = \varphi''(0) = \mu^2 + \mu \\
\alpha_3 &= E(X^3) = \varphi'''(0) = \mu^3 + 3\mu^2 + \mu \\
\alpha_4 &= E(X^4) = \varphi''''(0) = \mu^4 + 6\mu^3 + 7\mu^2 + \mu
\end{aligned}
$$

So, finally,

$$
\begin{aligned}
\mu_4 &= E\{(X-\mu)^4\} \\
&= E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 4\mu^3 E(X) + \mu^4 \\
&= \alpha_4 - 4\mu\alpha_3 + 6\mu^2\alpha_2 - 4\mu^3\alpha_1 + \mu^4 \\
&= (\mu^4 + 6\mu^3 + 7\mu^2 + \mu) - 4\mu(\mu^3 + 3\mu^2 + \mu) + 6\mu^2(\mu^2 + \mu) - 4\mu^3\mu + \mu^4 \\
&= 3\mu^2 + \mu
\end{aligned}
$$

and the asymptotic variance of V_n is

$$
\mu_4 - \mu_2^2 = 3\mu^2 + \mu - \mu_2 = 2\mu^2 + \mu
$$

So $\overline{X}_n$ has smaller asymptotic variance than V_n (for all values of μ) and the ARE is

$$
\frac{\mu}{\mu + 2\mu^2}
$$

Note that the ARE goes to zero as μ goes to infinity, so V_n gets arbitrarily bad for very large μ. Thus $\overline{X}_n$ is not only the more obvious method of moments estimator but also the better one.