Stat 5102 (Geyer) Spring 2016
Homework Assignment 7
Due Wednesday, November 2, 2016

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

7-1. Suppose $X_1, \ldots, X_n$ are IID Cauchy(μ, σ) and $\sigma = 1$ is known. We wish to do maximum likelihood estimation, which cannot be done in closed form, so you must use R. One needs a good estimate of the location parameter to use as a starting point for the optimization. The location parameter is the center of symmetry and also the median. Thus the sample median is a good starting point. Data for the problem are at the URL

`http://www.stat.umn.edu/geyer/5102/data/prob7-1.txt`

(a) Find the MLE for these data.

(b) Find the observed Fisher information evaluated at the MLE.

(c) Find an asymptotic 95% confidence interval for the parameter μ.

7-2. Suppose $x_1, \ldots, x_n$ are known numbers (not random), and we observe random variables $Y_1, \ldots, Y_n$ that are independent but *not* identically distributed random variables having distributions

$$Y_i \sim \mathcal{N}(\alpha + \beta x_i, \sigma^2),$$

where α, β, and σ^2 are unknown parameters.

(a) Write down the log likelihood for the parameters α, β, and $\varphi = \sigma^2$.

(b) Find the maximum likelihood estimates of these parameters.

(c) Find the expected Fisher information matrix for these parameters.

Caution: In taking expectations remember only the Y_i are random. The x_i are known constants.

7-3. Suppose $X_1, \ldots, X_n$ are IID Laplace(μ, σ) and both parameters are considered unknown. We found the log likelihood in problem 6-4. Find the MLE of μ and σ.
Hint: For each fixed value of σ show that the sample median is the MLE of μ, the argument being as in problem 6-13. Call that MLE $\hat{\mu}_n$, now plug this estimate into the log likelihood obtaining a function

$$\sigma \mapsto l_n(\hat{\mu}_n, \sigma) \qquad (*)$$

which is a twice differentiable function of σ. Find a point where the first derivative of $(*)$ is zero, and show that it is a local maximizer. That point is the MLE for σ.

7-4. Suppose $X_1, \ldots, X_n$ are IID Cauchy(μ, σ) and both parameters are unknown. We wish to do maximum likelihood estimation, which cannot be done in closed form, so you must use R. One needs a good estimate of both parameters to use as a starting point for the optimization. As in problem 7-1 the sample median is a good starting point for μ. In 5101 problem 8-10(c) we found that the interquartile range (IQR) of the standard Cauchy distribution is 2. Thus the IQR of the Cauchy(μ, σ) distribution is 2σ, and half the IQR is a good starting point for σ. The R function `IQR` estimates the IQR. Data for the problem are at the URL

`http://www.stat.umn.edu/geyer/5102/data/prob7-1.txt`

(a) Find the MLE vector for these data.

(b) Find the observed Fisher information matrix evaluated at the MLE.

(c) Find asymptotic 95% confidence intervals for the parameters μ and σ. Do not adjust to obtain 95% simultaneous coverage.

7-5. Suppose $X_1, \ldots, X_n$ are IID Gam(α, λ) and $\lambda = 1$ is known. Suppose $n = 50$, the MLE is $\hat{\alpha}_n = 2.73$ and we wish to do test of the hypotheses

$$\begin{aligned} H_0 &: \alpha = 2.5 \\ H_1 &: \alpha > 2.5 \end{aligned}$$

Find the asymptotic P-value for this test using the standardized MLE as the test statistic. You will have to use R to calculate Fisher information. Interpret the P-value.

Review Problems from Previous Tests

7-6. Suppose $X_1, \ldots, X_n$ are IID Gam(α, λ), where α is known and λ is unknown.

(a) Find the log likelihood for λ.

(b) Find the maximum likelihood estimate (MLE) for λ.

(c) Show that your MLE is the unique global maximizer of the log likelihood.

(d) Find the expected Fisher information for λ.

7-7. Suppose $X_1, \ldots, X_n$ are IID having PDF

$$f_\theta(x) = \theta x^{\theta - 1}, \qquad 0 < x < 1$$

where $\theta > 0$ is an unknown parameter.

(a) Find the log likelihood for θ.

(b) Find the maximum likelihood estimate (MLE) for θ.

(c) Find the asymptotic distribution of your MLE.

(d) Find an asymptotic 95% confidence interval for θ. (Hint: The 0.95 quantile of the standard normal distribution is 1.645, and the 0.975 quantile of the standard normal distribution is 1.96.)

7-8. Suppose X is $\text{Exp}(\lambda)$, where λ is an unknown parameter. We have only one observation.

(a) Describe how to do an exact (not approximate) test of the hypotheses

$$H_0 \colon \lambda = \lambda_0$$
$$H_1 \colon \lambda < \lambda_0$$

where λ_0 is a specified number (the value of λ hypothesized under H_0. Give a formula for the P-value of the test, an expression in terms of functions you can find on a calculator. (Hint: Consider the relationship between $E(X)$ and λ.)

(b) Calculate the P-value when $\lambda_0 = 1$ and $x = 3.7$.

(c) Interpret the P-value. What does it say about whether λ is larger or smaller than λ_0? What does it say about the statistical significance of this conclusion?