Stat 5101 First Midterm Exam **February 23, 2022**

Name ______________________ Student ID ______________

The exam is closed book and closed notes. You may use one $8\frac{1}{2} \times 11$ sheet of paper with formulas, etc. You may also use the handouts on "brand name distributions" and Greek letters. You may use a calculator. No other electronic devices are allowed.

Put all of your work on this test form (use the backs of pages if necessary). Show your work or give an explanation of your answer. No credit for numbers or formulas with no indication of where they came from. Leave no undone sums over the whole sample space or undone means or variances in your answers. But other than that requirement, there is no unique "correct" simplification for any answer. Any correct (and explained) answer gets full credit unless the question explicitly states otherwise.

Abbreviations used: independent and identically distributed (IID) and probability mass function (PMF).

The points for the questions total to 100. There are 6 pages and 5 problems.

1. [20 pts.] Suppose X is a random variable having PMF with parameter $\theta > 0$ given by

x	1	2	3	4
$f(x)$	$\frac{1}{2+2\theta}$	$\frac{1}{2+2\theta}$	$\frac{\theta}{2+2\theta}$	$\frac{\theta}{2+2\theta}$

In this problem simplify your answers so they do not leave undone a sum over the points in the sample space.

(a) Calculate $E_\theta(X)$.

(b) Calculate $E_\theta(X^2)$.

(c) Calculate $\text{var}_\theta(X)$.

2. [20 pts.] Suppose X is a random variable having distribution given by

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	$\frac{1+\theta}{45+9\theta}$	$\frac{2+\theta}{45+9\theta}$	$\frac{3+\theta}{45+9\theta}$	$\frac{4+\theta}{45+9\theta}$	$\frac{5+\theta}{45+9\theta}$	$\frac{6+\theta}{45+9\theta}$	$\frac{7+\theta}{45+9\theta}$	$\frac{8+\theta}{45+9\theta}$	$\frac{9+\theta}{45+9\theta}$

Find the PMF of the random variable $Y = g(X)$ where g is given by

x	-4	-3	-2	-1	0	1	2	3	4
$g(x)$	1	1	1	2	2	2	3	3	3

3. [20 pts.] Suppose $\mathbf{X}$ is a random vector with mean vector

$$\boldsymbol{\mu} = \begin{pmatrix} \mu \\ \mu \\ \mu \\ \mu \\ \mu \\ \mu \end{pmatrix}$$

and variance matrix

$$\mathbf{M} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & 0 & 0 & 0 \\ \gamma_1 & \gamma_0 & \gamma_1 & \gamma_2 & 0 & 0 \\ \gamma_2 & \gamma_1 & \gamma_0 & \gamma_1 & \gamma_2 & 0 \\ 0 & \gamma_2 & \gamma_1 & \gamma_0 & \gamma_1 & \gamma_2 \\ 0 & 0 & \gamma_2 & \gamma_1 & \gamma_0 & \gamma_1 \\ 0 & 0 & 0 & \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix}$$

where the parameters μ, γ_0, γ_1, and γ_2 satisfy are real numbers chosen to make $\mathbf{M}$ a positive semi-definite matrix (we cannot say more than that).

(a) Find $E(X_1 + X_2 + X_3 + X_4 + X_5 + X_6)$.

(b) Find $\text{var}(X_1 + X_2 + X_3 + X_4 + X_5 + X_6)$.

4. [20 pts.] Suppose X and Y are IID independent and identically distributed each having PMF f given by

x	0	1	2	3	4
$f(x)$	1/2	1/4	1/8	1/16	1/16

In each part give reasoning justifying your answer.

(a) Are $\sin(X)$ and $\cos(Y)$ independent?

(b) Are X and $X + Y$ independent?

(c) Are $X - Y$ and $X + Y$ independent?

5. [20 pts.] In this problem, simplify your answers so they do not contain any unevaluated binomial coefficients.

   (a) There are 3 red and 4 green balls in an urn, and we draw a random sample of size 5 with replacement from the urn (this means the balls are well mixed before each draw). If X is the number of red balls drawn, what is $\Pr(X \leq 1)$?

   (b) Exactly the same question as in part (a) except that we change with replacement to without replacement (in which case it does not matter whether the balls are well mixed between draws as long as they were well mixed before the first draw).