Stat 5101 (Geyer) Spring 2022
Homework Assignment 8
Due Wednesday, March 30, 2022

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

8-1. Let X have the standard Cauchy distribution, which has PDF defined in the brand name distributions handout

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1 + x^2}, \qquad -\infty < x < \infty.$$

(a) Find the quantile function for X.

(b) Find the median of X.

(c) Find the lower and upper quartiles of X.

Hint: the indefinite integral of $1/(1 + x^2)$ is arc tangent of x (inverse of the tangent function).

8-2. Suppose X has the $\text{Exp}(1)$ distribution.

(a) What is the best prediction of the value of X if minimizing expected squared error is the criterion?

(b) What is the best prediction of the value of X if minimizing expected absolute error is the criterion?

In this problem, we want numeric answers so we can see how different they are.

8-3. Suppose X has the $\text{Gam}(2, 1)$ distribution.

(a) What is the best prediction of the value of X if minimizing expected squared error is the criterion?

(b) What is the best prediction of the value of X if minimizing expected absolute error is the criterion?

In this problem, we want numeric answers so we can see how different they are. The answer to (b) can only be found using the computer.

8-4. Suppose the random vector (X, Y) has the PDF

$$f(x, y) = \frac{1}{3}(x + y + xy)e^{-x-y}, \qquad 0 < x < \infty,\ 0 < y < \infty. \tag{1}$$

Find the marginal PDF of X.

8-5. Suppose the random vector (X, Y) has the uniform distribution on the triangle

$$\{ (x, y) \in \mathbb{R}^2 : 0 < x < y < 1 \} \tag{2}$$

(a) Find the marginal PDF for X.

(b) Find the marginal PDF for Y.

Hint: be careful about ranges of integration; if y is fixed, then the range of x is $0 < x < y$; if x is fixed, then the range of y is $x < y < 1$.

8-6. Suppose the conditional distribution of Y given X has PDF

$$f(y \mid x) = \frac{6(x + xy + y^2)}{2 + 9x}, \qquad 0 < y < 1.$$

(a) Find $E(Y \mid x)$.

(b) Find $\operatorname{var}(Y \mid x)$.

8-7. Suppose the conditional distribution of Y given X is $\mathcal{N}(X, X)$.

(a) Find $E(Y \mid x)$.

(b) Find $\operatorname{var}(Y \mid x)$.

(c) Find $E(Y^2 \mid x)$.

8-8. Suppose the random vector (X, Y) has the PDF (1) Find the conditional PDF of Y given X.

8-9. Suppose the random vector (X, Y) has the uniform distribution on the triangle (2)

(a) Find the conditional PDF of Y given X.

(b) Find the conditional PDF of X given Y.

(c) Find the conditional mean of Y given X.

(d) Find the conditional mean of X given Y.

(e) Find the conditional median of Y given X.

(f) Find the conditional median of X given Y.

8-10. Suppose the random vector (X, Y) has the PDF

$$f(x, y) = \frac{4}{5}(x + y + xy), \qquad 0 < x < 1, \ 0 < y < 1.$$

(a) Find the conditional PDF of Y given X.

(b) Find the conditional mean of Y given X.

(c) Find the conditional median of Y given X.

8-11. Suppose the conditional distribution of Y given X is $\text{Bin}(n, X)$, where n is a known positive integer, and suppose the marginal distribution of X is $\text{Beta}(\alpha_1, \alpha_2)$. What is the conditional distribution of X given Y? Since this is a brand name distribution, no integrals need be done, it is enough to name the distribution and give its parameters as a function of Y, n, α_1, and α_2.

8-12. Suppose the conditional distribution of Y given X is $\mathcal{N}(\mu, 1/X)$, where μ is a known real number, and suppose the marginal distribution of X is $\text{Gam}(\alpha, \lambda)$. What is the conditional distribution of X given Y? Since this is a brand name distribution, no integrals need be done, it is enough to name the distribution and give its parameters as a function of Y, μ, α, and λ.

Review Problems from Previous Tests

8-13. Suppose X is a random variable having probability density function (PDF) given by

$$f(x) = \frac{6}{5}(x + x^2), \qquad 0 < x < 1.$$

(a) Calculate $E(X)$.

(b) Calculate $\text{var}(X)$.

8-14. Suppose X is a random variable having PDF given by

$$f(x) = \frac{2}{x^3}, \qquad 1 < x < \infty.$$

Find the PDF of the random variable $Y = \log(X)$.

8-15. Calculate the PDF corresponding to the DF

$$F(x) = \begin{cases} 0, & x \le 0 \\ (x + x^2)/2, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$$

8-16. Calculate the DF corresponding to the PDF

$$f(x) = \frac{\cos(x)}{2}, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Define the DF on the whole real line.

8-17. Suppose the random variable X has the PDF (not the DF) given by

$$f(x) = 2x, \qquad 0 < x < 1.$$

Find the median (not the mean) of the distribution of X.

8-18. Suppose the random vector (X, Y) has the PDF

$$f(x, y) = \frac{1}{2}(x + y + xy)e^{-x}, \qquad 0 < x < \infty,\ 0 < y < 1. \tag{3}$$

Find the marginal PDF of Y. The definition of a function describes the domain as well as the rule.

8-19. Suppose the random vector (X, Y) has the PDF

$$f(x, y) = \frac{6}{7}(x + y)^2, \qquad 0 < x < 1,\ 0 < y < 1.$$

Find the conditional PDF of Y given X. The definition of a function describes the domain as well as the rule.

8-20. Suppose the conditional distribution of Y given X is $\text{Gam}(\beta, X)$, where β is a known real number, and suppose the marginal distribution of X is $\text{Gam}(\alpha, \lambda)$. What is the conditional distribution of X given Y? Since this is a brand name distribution, no integrals need be done, it is enough to name the distribution and give its parameters as a function of Y, α, β, and λ.