

Stat 5101 (Geyer) Spring 2022  
Homework Assignment 6  
Due Wednesday, March 16, 2022

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

**6-1.** The function

$$f(x) = 2x, \quad 0 < x < 1$$

is a PDF. Suppose  $X$  is a random variable having this PDF.

- (a) Calculate  $E(X)$ .
- (b) Calculate  $E(X^2)$ .
- (c) Calculate  $\text{var}(X)$ .

**6-2.** The function

$$f(x, y) = x + y, \quad 0 < x < 1, 0 < y < 1$$

is a PDF. Suppose  $(X, Y)$  is a random vector having this PDF.

- (a) Calculate  $E(X)$ .
- (b) Calculate  $\text{var}(X)$ .
- (c) Calculate  $\text{cov}(X, Y)$ .

(By symmetry,  $E(Y) = E(X)$  and  $\text{var}(Y) = \text{var}(X)$  so we do not need to calculate them.)

**6-3.** Suppose  $(X, Y)$  is a continuous random vector having PDF  $f$ . Say for each of the following definitions of  $f$  whether  $X$  and  $Y$  are independent or not.

- (a)  $f(x, y) = 4xy, 0 < x < 1, 0 < y < 1$ .
- (b)  $f(x, y) = 8xy, 0 < x < y < 1$ .
- (c)  $f(x, y) = 144(x - 1/2)^2(y - 1/2)^2, 0 < x < 1, 0 < y < 1$ .
- (d)  $f(x, y) = 288(x - 1/2)^2(y - 1/2)^2, 0 < x < y < 1$ .

**6-4.** Suppose  $X$  is a continuous random variable having PDF

$$f(x) = \begin{cases} 1+x, & -1 \leq x < 0 \\ 1-x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

- (a) Find  $E(X)$ .
- (b) Find  $E(X^2)$ .
- (c) Find  $\text{var}(X^2)$ .

Hint: Since the PDF has a case-splitting formula, you must split integrals into pieces

$$E\{g(X)\} = \int_{-1}^0 g(x)f(x) dx + \int_0^1 g(x)f(x) dx$$

such that the PDF is defined by one formula for each piece.

**6-5.** Suppose  $X$  is a continuous random variable having the  $\text{Exp}(\lambda)$  distribution. Write  $\mu = E(X)$ .

- (a) Find  $E\{(X - \mu)^3\}$ .
- (b) Find  $E\{(X - \mu)^4\}$ .

**6-6.** Suppose  $(X, Y)$  is a continuous random vector having PDF

$$f(x, y) = 2, \quad 0 < x < y < 1$$

- (a) Find  $E(X)$
- (b) Find  $E(Y)$
- (c) Find  $E(X^2)$
- (d) Find  $E(Y^2)$
- (e) Find  $E(XY)$
- (f) Find  $\text{var}(X)$
- (g) Find  $\text{var}(Y)$
- (h) Find  $\text{cov}(X, Y)$

Hint: the limits of integration are a bit tricky.

$$E\{g(X, Y)\} = 2 \int_0^1 \int_0^y g(x, y) dx dy = 2 \int_0^1 \int_x^1 g(x, y) dy dx$$

**6-7.** Suppose  $X$  is a continuous random variable having the same PDF as in problem 6-4. Find its distribution function. Be sure to define the DF on the whole real line.

**6-8.** Suppose  $X$  is a continuous random variable having DF

$$F(x) = \begin{cases} 0, & x \leq 1 \\ 1 - 1/x, & x > 1 \end{cases}$$

Find its PDF. Define the PDF on the whole real line.

**6-9.** Suppose  $U$  has the Unif(0, 1) distribution. What is the PDF of

$$Y = -\frac{1}{\lambda} \log(U)$$

**6-10.** Suppose  $X$  has the Unif(-1, 1) distribution. What is the PDF of

$$Y = X^2$$

**6-11.** Suppose  $(X, Y)$  has the uniform distribution on the disk

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

What is the PDF of the random vector  $(R, T)$  which is  $(X, Y)$  expressed in polar coordinates?

Hint: The map from  $(r, t)$  to  $(x, y)$  is given by

$$\begin{aligned} x &= r \cos(t) \\ y &= r \sin(t) \end{aligned}$$